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The topological structure of vortex in BEC

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Abstract. We find that there exists an elementary topological current in Bose-Einstein condensation. Based on the ϕ-mapping topological current theory, the implicit function theorem and the Taylor expansion, the topological structure of vortex lines is detailed in the neighborhoods of the bifurcation points of the condensate wave function.

PACS. 03.75.Fi Phase coherent atomic ensemble (Bose condensation) – 02.40.Pc General topology

1 Introduction

An important consequence of quantum statistics is that, below some critical temperature, bosons are predicted to the ground state [1]. This macroscopic quantum phenomenon, termed Bose-Einstein condensation (BEC), has recently been observed in a dilute atomic vapor [2,3]. Vortex states which are well-known in superfluid and superconductor also play an important role in characterizing the superfluid properties of BEC systems [4–6]. In this paper, it is found that in BEC, the vorticity is just the topological current constructed by the condensate wave function. Using the ϕ-mapping topological current theory [7], we obtain the intrinsic relation between vorticity and condensate wave function as well as the topological structure of vorticity. We also found that there exists the crucial case of branch process. The vortex lines cross, split or merge at the bifurcation points.

2 The intrinsic relation between vorticity and condensate wave function

Several important conclusions about the properties of BEC can be drawn simply from the existence of a macroscopic quantity, the condensate wave function Ψ. The current $\mathbf{j}_0$ is characterized by the condensate wave function Ψ [8]:

$$\mathbf{j}_0 = -i\hbar(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)/2$$

i.e.

$$\mathbf{j}_0 = m \mid \Psi \mid^2 \mathbf{V}, \tag{1}$$

where the definition of velocity $\mathbf{V}$ is

$$\mathbf{V} = -\frac{i\hbar}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)/ \mid \Psi \mid^2 . \tag{2}$$

If we denote the wave function

$$\Psi = \mid \Psi \mid e^{i\Theta}, \tag{3}$$

the velocity $\mathbf{V}$ can be written as

$$\mathbf{V} = \frac{\hbar}{m}\nabla\Theta \tag{4}$$

from which the definition of the velocity potential Θ came.

It is well-known that the condensate wave function $\Psi(\mathbf{x})$, which is like the Schrödinger wave function, can be looked upon as a section of a complex line bundle with base manifold R^3 [9]. Denote the wave function $\Psi(\mathbf{x})$ as

$$\Psi(\mathbf{x}) = \Phi^1(\mathbf{x}) + i\Phi^2(\mathbf{x}), \tag{5}$$

where $\Phi^1(\mathbf{x})$ and $\Phi^2(\mathbf{x})$ are two components of a two dimensional vector field

$$\boldsymbol{\Phi} = (\Phi^1, \Phi^2) \tag{6}$$

on R^3. In our viewpoints, the topology of BEC should be determined by the intrinsic topology character of the section of this line bundle. From equation (2) one can prove that

$$\mathbf{V} = -\frac{\hbar}{m}\epsilon_{ab}\nabla n^a n^b = -\frac{\hbar}{m}\epsilon_{ab}\partial_k n^a n^b \mathbf{e}_k$$

and the vorticity

$$\nabla \times \mathbf{V} = \frac{\hbar}{m}\mathbf{e}_i(\epsilon^{ijk}\epsilon_{ab}\partial_j n^a \partial_k n^b) \tag{7}$$

where $\mathbf{e}_k$ $(k = 1, 2, 3)$ are the base vectors in Cartesian coordinate system, and n^a is a two dimensional unit vector field

$$\mathbf{n} = \boldsymbol{\Phi}/ \parallel \Phi \parallel, \tag{8}$$

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where

$$\| \Phi \|^2 = \Phi^a \Phi^a = | \Psi |^2 .$$

From equation (8), it is easy to see that the zeroes of the wave function $\Psi(\mathbf{x})$ are just the singularities of $\mathbf{n}(\mathbf{x})$. Using the two dimensional unit vector field (8), one can construct a topological current of wave function:

$$J^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b,$$

which is the special case of the ϕ-mapping topological current theory [7]. One can see that

$$\partial_i J^i = 0, \qquad (9)$$

which is the continuity equation for the condensate. So that equation (7) is turned into

$$\nabla \times \mathbf{V} = \frac{h}{m} J^i \mathbf{e}_i. \qquad (10)$$

Therefore in BEC the vorticity $\nabla \times \mathbf{V}$ can be expressed in terms of the topological current of wave function. Using equation (8) and

$$\partial_i n^a = \frac{\partial_i \Phi^a}{\|\Phi\|} - \frac{\Phi^a \partial_i \|\Phi\|}{\|\Phi\|^2},$$
$$\frac{\partial}{\partial \Phi^a} \ln \|\Phi\| = \frac{\Phi^a}{\|\Phi\|^2},$$

J^i is changed into

$$J^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \frac{\partial}{\partial \Phi^c} \frac{\partial}{\partial \Phi^a} \ln \|\Phi\| \partial_j \Phi^a \partial_k \Phi^b.$$

By defining the vector Jacobian of $\boldsymbol{\Phi}$:

$$D^i \left(\frac{\Phi}{x} \right) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \Phi^a \partial_k \Phi^b, \qquad (11)$$

and making use of Laplacian relation in ϕ space

$$\frac{\partial}{\partial \Phi^a} \frac{\partial}{\partial \Phi^a} \ln \|\Phi\| = 2\pi \delta^2(\boldsymbol{\Phi}),$$

we do obtain the δ like topological current

$$J^i = \delta^2(\boldsymbol{\Phi}) D^i \left(\frac{\Phi}{x} \right). \qquad (12)$$

Thus we have the important relation between vorticity and condensate wave function in BEC:

$$\nabla \times \mathbf{V} = \frac{h}{m} \delta^2(\boldsymbol{\Phi}) \mathbf{D} \left(\frac{\Phi}{x} \right), \qquad (13)$$

where

$$\mathbf{D} \left(\frac{\Phi}{x} \right) = D^i \left(\frac{\Phi}{x} \right) \mathbf{e}_i. \qquad (14)$$

From equation (13), we see that the vorticity $\nabla \times \mathbf{V}$ does not vanish at the zero points of $\boldsymbol{\Phi}$, *i.e.*

$$\Phi^1(x, y, z) = 0, \quad \Phi^2(x, y, z) = 0. \qquad (15)$$

The solutions of equation (15) are generally expressed as

$$x = x_i(l), \quad y = y_i(l), \quad z = z_i(l) \quad i = 1, 2, \ldots, N \quad (16)$$

which represent N singular string $L_i (i = 1, 2, \ldots, N)$ where $\boldsymbol{\Phi} = 0$ in space. The location and the direction of the ith vortex are determined by the ith singular string L_i and the vector Jacobian $\mathbf{D}(\Phi/x)$ on L_i respectively. When the vector field $\boldsymbol{\Phi}$, *i.e.*, the wave function Ψ, has no zero values, $\delta^2(\boldsymbol{\Phi})$ is zero and equation (13) becomes

$$\nabla \times \mathbf{V} = 0, \qquad (17)$$

which is the condition of irrotationality. So, equation (13) describes both the vortex-state and the irrotationality-state.

In the theory of δ function of the function $\boldsymbol{\Phi}(\mathbf{x})$ [10,11], one can prove that

$$\delta^2(\boldsymbol{\Phi}) = \sum_{i=1}^{N} \beta_i \eta_i \int_{L_i} \frac{\delta^3(\mathbf{x} - \mathbf{x}_i(l))}{D\left(\frac{\Phi}{u}\right)_{\Sigma_i}} dl, \qquad (18)$$

where

$$D \left(\frac{\Phi}{u} \right)_{\Sigma i} = \left(\frac{1}{2} \epsilon^{jk} \epsilon_{mn} \frac{\partial \Phi^m}{\partial u^j} \frac{\partial \Phi^n}{\partial u^k} \right). \qquad (19)$$

L_i is the ith singular string where $\Psi = 0$ and Σ_i is a planar element where $u = (u^1, u^2)$ are the intrinsic coordinates. We stress that Σ_i is normal to L_i at point $\mathbf{x}_i(l)$. The positive integer β_i is the Hopf index of ϕ mapping and

$$\eta_i = \operatorname{sgn} \left[D \left(\frac{\Phi}{u} \right)_{\Sigma_i} \right] \qquad (20)$$

is the Brouwer degree of ϕ mapping. The meaning of β_i is that when the point $\mathbf{x}$ covers the neighborhood of the zero $\mathbf{x}_i$ on Σ_i once, the vector field $\boldsymbol{\Phi}$ covers the corresponding region β_i times. Using equations (14, 19) as well as following the ϕ-mapping topological current theory, we can prove

$$\left[\mathbf{D} \left(\frac{\Phi}{x} \right) \Big/ D \left(\frac{\Phi}{u} \right)_{\Sigma_i} \right]_{\mathbf{x}_i(l)} = \frac{d\mathbf{x}_i}{dl}, \qquad (21)$$

then from equations (18, 21) we have

$$\delta^2(\boldsymbol{\Phi}) \mathbf{D} \left(\frac{\Phi}{x} \right) = \sum_{i=1}^{N} \beta_i \eta_i \int_{L_i} d\mathbf{x}_i \delta^3(\mathbf{x} - \mathbf{x}_i). \qquad (22)$$

Direct substitution of equation (22) into equation (13) leads to the topological structure of vorticity:

$$\nabla \times \mathbf{V} = \frac{h}{m} \sum_{i=1}^{N} \beta_i \eta_i \int_{L_i} d\mathbf{x}_i \delta^3(\mathbf{x} - \mathbf{x}_i). \qquad (23)$$

It is obvious to see that equation (23) represents N isolated vortices of which the ith vortex is charged with the topological charge $\beta_i \eta_i$.

3 Bifurcation of vortex line

The above discussions are based on the condition that only one of the Jacobians $D^j\,(\Phi/x)$ $(j=1,2,3)$ can be zero at some points along L_i. When two of them are zero at some points along L_i, it is shown that there exist the crucial cases of branch process. We call these points bifurcation points, which are determined by

$$\begin{aligned}\Phi^1(x,y,z)=0,\\ \Phi^2(x,y,z)=0,\end{aligned}\tag{24}$$

and

$$D^3\left(\frac{\Phi}{x}\right)=0,\qquad D^1\left(\frac{\Phi}{x}\right)=0.\tag{25}$$

We denote one of the bifurcation points along L_i as $\mathbf{x}_i^*=\mathbf{x}_i(l^*)=(x_i^*,y_i^*,z_i^*)$. These two restrictive conditions (25) will lead to an important fact that the functional relationship between z and x is not unique in the neighborhood of $\mathbf{x}_i^*$. In our vorticity topological current theory, this fact is easily seen from

$$\frac{dx}{dz}=\left.\frac{D^1\left(\frac{\Phi}{x}\right)}{D^3\left(\frac{\Phi}{x}\right)}\right|_{\mathbf{x}_i^*}\tag{26}$$

which under equation (25) directly shows that the direction of the integral curve of equation (26) is indefinite at $\mathbf{x}_i^*$. Therefore the very point $\mathbf{x}_i^*$ is called a bifurcation point of the condensate wave function. With the aim of finding the different directions of all branch curves at the bifurcation point, we suppose that

$$\left.\frac{\partial\Phi^1}{\partial y}\right|_{\mathbf{x}_i^*}\neq 0.\tag{27}$$

From $\Phi^1(x,y,z)=0$, the implicit function theorem says that there exists one and only one functional relationship

$$y=y(x,z).\tag{28}$$

Substituting equation (28) into Φ^1, we have

$$\Phi^1(x,y(x,z),z)\equiv 0$$

which gives

$$\begin{aligned}
&\frac{\partial\Phi^1}{\partial y}f_x^y=-\frac{\partial\Phi^1}{\partial x},\\
&\frac{\partial\Phi^1}{\partial y}f_z^y=-\frac{\partial\Phi^1}{\partial z},\\
&\frac{\partial\Phi^1}{\partial y}f_{xx}^y=-2\frac{\partial^2\Phi^1}{\partial y\partial x}f_x^y-\frac{\partial^2\Phi^1}{\partial y^2}(f_x^y)^2-\frac{\partial^2\Phi^1}{\partial x^2},\\
&\frac{\partial\Phi^1}{\partial y}f_{xz}^y=-\frac{\partial^2\Phi^1}{\partial y\partial z}f_x^y-\frac{\partial^2\Phi^1}{\partial y\partial x}f_z^y-\frac{\partial^2\Phi^1}{\partial y^2}f_z^yf_x^y-\frac{\partial^2\Phi^1}{\partial x\partial z},\\
&\frac{\partial\Phi^1}{\partial y}f_{zz}^y=-2\frac{\partial^2\Phi^1}{\partial y\partial z}f_z^y-\frac{\partial^2\Phi^1}{\partial y^2}(f_z^y)^2-\frac{\partial^2\Phi^1}{\partial z^2},
\end{aligned}\tag{29}$$

where the partial derivatives are

$$f_x^y=\frac{\partial y}{\partial x},\qquad f_z^y=\frac{\partial y}{\partial z},\qquad f_{xx}^y=\frac{\partial^2 y}{\partial x^2},$$
$$f_{xz}^y=\frac{\partial^2 y}{\partial x\partial z},\qquad f_{zz}^y=\frac{\partial^2 y}{\partial z^2}\,.$$

From these expressions the values of $f_x^y, f_z^y, f_{xx}^y, f_{xz}^y$ and f_{zz}^y at $\mathbf{x}_i^*$ can be calculated.

In order to explore the behavior of vortex lines at the bifurcation points, let us investigate the Taylor expansion of

$$F(x,z)=\Phi^2(x,y(x,z),z)\tag{30}$$

in the neighborhood of $\mathbf{x}_i^*$, which according to equation (24) must vanish at the bifurcation point, *i.e.*

$$F(\mathbf{x}_i^*)=0.\tag{31}$$

From equation (30), the first order partial derivatives of $F(x,z)$ with respect to x and z can be expressed by

$$\begin{aligned}
\frac{\partial F}{\partial x}&=\frac{\partial\Phi^2}{\partial x}+\frac{\partial\Phi^2}{\partial y}f_x^y,\\
\frac{\partial F}{\partial z}&=\frac{\partial\Phi^2}{\partial z}+\frac{\partial\Phi^2}{\partial y}f_z^y.
\end{aligned}\tag{32}$$

By making use of equations (29, 32) and Cramer's rule, we can prove that the two restrictive conditions (25) can be rewritten as

$$\begin{aligned}
D^3\left(\frac{\Phi}{x}\right)\Big|_{\mathbf{x}_i^*}&=\left(\frac{\partial F}{\partial x}\frac{\partial\Phi^1}{\partial y}\right)\Big|_{\mathbf{x}_i^*}=0,\\
D^1\left(\frac{\Phi}{x}\right)\Big|_{\mathbf{x}_i^*}&=\left(\frac{\partial F}{\partial z}\frac{\partial\Phi^1}{\partial y}\right)\Big|_{\mathbf{x}_i^*}=0,
\end{aligned}$$

which give

$$\left.\frac{\partial F}{\partial x}\right|_{\mathbf{x}_i^*}=0,\qquad \left.\frac{\partial F}{\partial z}\right|_{\mathbf{x}_i^*}=0\tag{33}$$

by considering equation (27). The second order partial derivatives of the function F are found out to be

$$\begin{aligned}
\frac{\partial^2F}{\partial x^2}&=\frac{\partial^2\Phi^2}{\partial x^2}+2\frac{\partial^2\Phi^2}{\partial y\partial x}f_x^y+\frac{\partial\Phi^2}{\partial y}f_{xx}^y+\frac{\partial^2\Phi^2}{\partial y^2}(f_x^y)^2\\
\frac{\partial^2F}{\partial x\partial z}&=\frac{\partial^2\Phi^2}{\partial x\partial z}+\frac{\partial^2\Phi^2}{\partial y\partial x}f_z^y+\frac{\partial^2\Phi^2}{\partial y\partial z}f_x^y\\
&\quad+\frac{\partial\Phi^2}{\partial y}f_{xz}^y+\frac{\partial^2\Phi^2}{\partial y^2}f_x^yf_z^y\\
\frac{\partial^2F}{\partial z^2}&=\frac{\partial^2\Phi^2}{\partial z^2}+2\frac{\partial^2\Phi^2}{\partial y\partial z}f_z^y+\frac{\partial\Phi^2}{\partial y}f_{zz}^y+\frac{\partial^2\Phi^2}{\partial y^2}(f_z^y)^2
\end{aligned}$$

which at $\mathbf{x}_i^*$ are denoted by

$$A=\left.\frac{\partial^2F}{\partial x^2}\right|_{\mathbf{x}_i^*},\qquad B=\left.\frac{\partial^2F}{\partial x\partial z}\right|_{\mathbf{x}_i^*},\qquad C=\left.\frac{\partial^2F}{\partial z^2}\right|_{\mathbf{x}_i^*}.\tag{34}$$

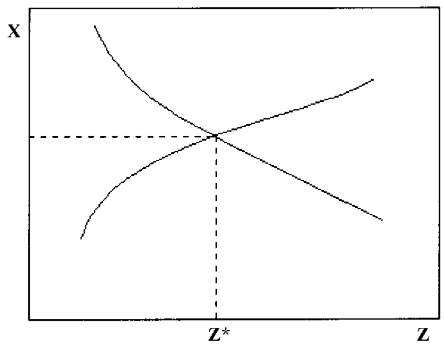

Fig. 1. Bifurcation solution for equation (37): two branch curves intersect with different directions at the bifurcation point.

Then, from equations (31, 33, 34), we obtain the Taylor expansion of $F(x,z)$ at $\mathbf{x}_i^*$:

$$F(x,z) = \frac{1}{2}A(x-x_i^*)^2 + B(x-x_i^*)(z-z_i^*) + \frac{1}{2}C(z-z_i^*)^2$$

which by equation (30) is the behavior of Φ^2 in the neighborhood of $\mathbf{x}_i^*$. Because of the second equation of (24), we get

$$A(x-x_i^*)^2 + 2B(x-x_i^*)(z-z_i^*) + C(z-z_i^*)^2 = 0$$

which leads to

$$A\left(\frac{dx}{dz}\right)^2 + 2B\frac{dx}{dz} + C = 0 \tag{35}$$

and

$$C\left(\frac{dz}{dx}\right)^2 + 2B\frac{dz}{dx} + A = 0. \tag{36}$$

The solutions of equation (35) or (36) give different directions of the branch curves at the bifurcation point. There are four possible cases.

- *Case 1* ($A \neq 0$): for $\Delta = 4(B^2 - AC) > 0$, from equation (35) we get two different directions

$$\left.\frac{dx}{dz}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}, \tag{37}$$

  which is shown in Figure 1, where two branch curves intersect with different directions.
- *Case 2* ($A \neq 0$): for $\Delta = 4(B^2 - AC) = 0$, from equation (35) we get only one direction

$$\left.\frac{dx}{dz}\right|_{1,2} = -\frac{B}{A}. \tag{38}$$

  which includes three important cases. One, two branch curves tangentially contact (see Fig. 2a). Two, two curves merge into one curve (see Fig. 2b). Three, one curve resolves into two curves (see Fig. 2c).

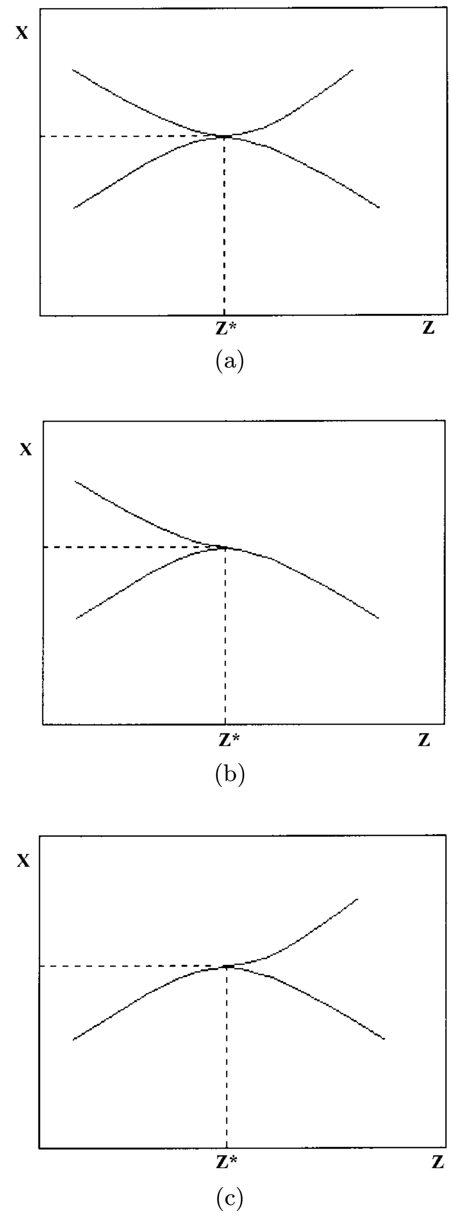

Fig. 2. Bifurcation solutions for equation (38). Vortex lines have the same direction when they cross. (a) Two branch curves tangentially contact at the bifurcation point, *i.e.* two vortex lines tangentially contact at the bifurcation point. (b) Two curves merge into one curve at the bifurcation point, *i.e.* two vortex lines merge into one vortex line at the bifurcation point. (c) One curve resolves into two curves at the bifurcation point, *i.e.* one vortex line splits into two vortex lines at the bifurcation point.

- *Case 3* ($A = 0$, $C \neq 0$): for $\Delta = 4(B^2 - AC) = 0$, from equation (36) we have

$$\left.\frac{dz}{dx}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0, \quad -\frac{2B}{C}. \tag{39}$$

  This case is shown in Figure 3.

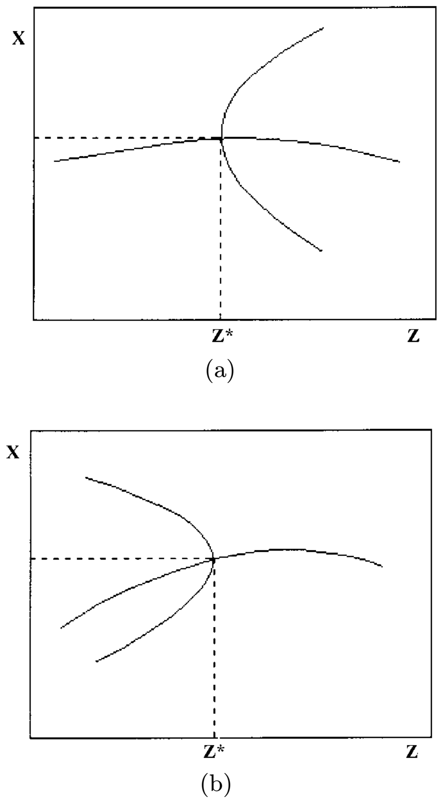

Fig. 3. Two important cases of equation (39). (a) One vortex line splits into three vortex lines at the bifurcation point. (b) Three vortex lines merge into one vortex line at the bifurcation point.

- *Case 4* ($A = C = 0$): equations (35, 36) gives respectively

$$\frac{dx}{dz} = 0, \qquad \frac{dz}{dx} = 0. \tag{40}$$

This case shows that two curves intersect normally at the bifurcation point, which is similar to *case 3*: (a) three vortex lines merge into one vortex line at the bifurcation point; (b) one vortex line splits into three vortex lines at the bifurcation point.

The remainder component dy/dz can be given by

$$\frac{dy}{dz} = f_x^y \frac{dx}{dz} + f_z^y$$

where partial derivative coefficients f_x^y and f_z^y have been calculated in equation (29).

Now, the topological structure of vortex lines is detailed in the neighborhoods of the bifurcation points of the condensate wave function. Besides the crossing of vortex lines, *i.e.* two vortex lines cross at the bifurcation point (see Figs. 1 and 2a), splitting and merging of vortex lines are also included. When a multicharged vortex line pass the bifurcation point, it may split into several vortex lines along different branch curves (see Figs. 2c and 3a). On the contrary, several vortex lines can merge into one vortex line at the bifurcation point (see Figs. 2b and 3b). The continuity equation for the condensate (9) shows the sum of the topological charge of final vortex line(s) must be equal to that of the initial vortex line(s) at the bifurcation point, *i.e.*

$$\sum_f \beta_{l_f} \eta_{l_f} = \sum_i \beta_{l_i} \eta_{l_i} \tag{41}$$

for fixed l.

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